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PROOF OF FIRST STANDARD FORM OF NONELEMENTARY FUNCTIONS

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ABSTRACT

In the present paper we have proved one of the six standard forms of indefinite nonintegrable functions (classically known as nonelementary functions) and their examples given by Yadav & Sen by applying strong Liouville's theorem, its special case, some well-known nonelementary functions and two properties due to Marchisotto & Zakeri.

Key Words: Nonelementary functions, strong Liouville's theorem etc.

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I INTRODUCTION

A natural query arises in calculus that "*what type of functions cannot be integrated?*" or "*which indefinite integrals are not elementary*?" The first example which leads us beyond the region of elementary functions is the *elliptic integrals* due to *John Wallis* (1655). Such integrals cannot be evaluated in terms of the elementary functions was proved by *Joseph Liouville* in 1833 and the main results on functions with nonelementary integrals began with *Strong Liouville Theorem (1835)* and *Strong Liouville Theorem (special case, 1835)*. *Marchisotto & Zakeri* (1994) studied nonelementary functions and mentioned two important examples 4 and 5 [5, pp.300-301], which are treated as properties in proving the functions elementary and nonelementary. By applying the above properties we get the following well-known nonelementary functions:

$$\int e^{x^2}dx\,,\ \int e^{ax^2}dx,(a\neq 0)\,,\ \int e^{-x^2}dx\,,\ \int\frac{e^{-x}}{x}dx\ ,\ \int\frac{e^{x}}{x}dx\,,\ \int\frac{\sin x}{x}dx\,,\ \int\frac{\cos x}{x}dx$$

Proofs of Nonelementary Functions: *Yadav & Sen* [6] have given six standard forms of indefinite nonintegrable functions out of which form-1 is as follows:

"An indefinite integral of the form $\int \frac{e^{f(x)}}{f'(x)}dx$ *, where f(x) is a polynomial function of degree ≥2, or a trigonometric (not inverse trigonometric) function, or a hyperbolic (not inverse hyperbolic) function is always nonintegrable i. e. nonelementary."*

Proof: We will prove it taking different possible cases as follows:

Case I: When f(x) is an algebraic function (polynomial) of degree ≥2:

We have $\int \frac{e^{f(x)}}{f'(x)}dx = \int g(x)e^{f(x)}dx$, [Taking $g(x) = \frac{1}{f'(x)}$]. From strong Liouville theorem (special case), $\int g(x)e^{f(x)}dx$ is elementary if and only if there exists a rational function R(x) such that

$$g(x) = R'(x) + R(x)f'(x) \Rightarrow \frac{1}{f'(x)} = R'(x) + R(x)f'(x)$$

Let $R(x) = \frac{p(x)}{q(x)}$, where g.c.d.(p(x), q(x))=1. Then we have

$$\frac{1}{f'(x)} = R'(x) + R(x)f'(x)$$

$$\Rightarrow f'(x)q(x)p'(x) - f'(x)p(x)q'(x) + [f'(x)]^2 p(x)q(x) = [q(x)]^2 \ \ldots (1.1)$$

$$\Rightarrow f'(x)p'(x) - q(x) + [f'(x)]^2 p(x) = \frac{f'(x)p(x)q'(x)}{q(x)}$$

Which implies q(x)|f'(x) as q(x) cannot divide p(x) and q'(x). In this case either q(x)=k, a constant or a polynomial of degree less than or equal to the degree of f'(x).

For q(x)=k, from (1.1) we have $f'(x)kp'(x) + [f'(x)]^2 p(x)k = k^2$ …………… (1.2)

Comparing degrees of x in (1.2) results out in a contradiction. Hence q(x) cannot be a constant.

For q(x) a polynomial of degree less than or equal to the degree of f'(x), we have since q(x)|f'(x), let us assume that f'(x)=q(x).h(x). Then from (1.1)

$$q(x)h(x)q(x)p'(x) - q(x)h(x)p(x)q'(x) + [q(x)h(x)]^2 p(x)q(x) = [q(x)]^2$$

$$\Rightarrow h(x)p'(x) - 1 + q(x)[h(x)]^2 p(x) = \frac{h(x)p(x)q'(x)}{q(x)} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3)$$

Which implies q(x)|h(x), since q(x) cannot divide p(x) and q'(x). Let h(x)=q(x).ξ(x). Then from (1.3), we have

$$q(x)\xi(x)p'(x)-p(x)q'(x)\xi(x)+[q(x)]^3[\xi(x)]^2p(x)=1 \quad \text{.................} (1.4)$$

Comparing the degrees of x in both sides in (1.4) results out in a contradiction. Therefore such R(x) does not exist, i. e., the given function is nonelementary.

Case II: **When f(x) be a trigonometric (not inverse trigonometric) function:**

Let us consider them one by one.

1.1 For sine function, we have

$$\int \frac{e^{f(x)}dx}{f'(x)} = \int \frac{e^{\sin\varphi(x)}dx}{\varphi'(x)\cos\varphi(x)},$$

where φ(x) be any polynomial of degree ≥1. On putting sinφ(x)=z, we have

$$\int \frac{e^{\sin\varphi(x)}dx}{\varphi'(x)\cos\varphi(x)} = \int \frac{e^z dz}{[\varphi'(x)]^2(1-z^2)} \quad \text{..................} (1.1.1)$$

Sub-case I: When φ(x) is linear, let φ(x)=x+b. Then from (1.1.1) we have

$$\int \frac{e^z dz}{[\varphi'(x)]^2(1-z^2)} = \int \frac{e^z dz}{(1-z^2)} = \frac{1}{2}\left[\int \frac{e^z dz}{(1-z)} + \int \frac{e^z dz}{(1+z)}\right],$$

where

$$\int \frac{e^z dz}{(1-z)} = -e\int \frac{e^{-p}}{p}dp, [\text{Putting } 1-z=p]$$

and $$\int \frac{e^z dz}{(1+z)} = \frac{1}{e}\int \frac{e^p}{p}dp, [\text{Putting } 1+z=p].$$

Both are nonelementary from example-4 due to Marchisotto et al [5, pp.300]. Therefore the given function is also nonelementary.

Sub-case II: When φ(x) is a polynomial of degree 2. Let us consider φ(x)=x^2+bx+c. Then we have from (1.1.1), on putting sinφ(x)=z

$$\int \frac{e^z dz}{[\varphi'(x)]^2(1-z^2)} = \frac{1}{4}\int \frac{e^z dz}{[\sin^{-1}z+k](1-z^2)}, \text{where } k=\frac{b^2-4c}{4}$$

$$= \frac{1}{4}\int \frac{e^z dz}{[\sin^{-1}z+k]\sqrt{(1-z^2)}\sqrt{(1-z^2)}}$$

$$= \int F\left[z, e^z, \sqrt{1-z^2}, \sin^{-1}z\right]dz = \int F[z, y_1, y_2, y_3]dz$$

$$\left[\frac{dy_1}{dz}=e^z=y_1,\frac{dy_2}{dz}=\frac{-z}{\sqrt{1-z^2}}=\frac{-z}{y_2},\frac{dy_3}{dz}=\frac{1}{\sqrt{1-z^2}}=\frac{1}{y_2}\right]$$

Applying strong Liouville theorem, part (b), we find that it is elementary if and only if there exists an identity of the form

$$\frac{e^z}{4[\sin^{-1}z+k](1-z^2)}=\frac{d}{dz}\left[U_0+\sum_{i=1}^{n}c_i\log U_i\right]$$

$$\Rightarrow\frac{e^z}{4[\sin^{-1}z+k](1-z^2)}=\left[\frac{dU_0}{dz}+\sum_{i=1}^{n}c_i\frac{U'_i}{U_i}\right],$$

where each U_j is a function of z, y_1, y_2, and y_3. Considering different forms of U_j like

$$\log[(\sin^{-1}z+k)e^z],e^z\log(\sin^{-1}z+k)$$

we find that no such U_j exist. Hence the given function is nonelementary. Similarly we can prove it nonelementary for higher degree polynomials φ(x).

1.2 For cosine function, we have

$$\int\frac{e^{f(x)}dx}{f'(x)}=\int\frac{e^{\cos\varphi(x)}dx}{-\varphi'(x)\sin\varphi(x)},$$

where φ(x) be any polynomial of degree ≥1. On putting cosφ(x)=z, we have

$$\int\frac{e^{\cos\varphi(x)}dx}{-\varphi'(x)\sin\varphi(x)}=\int\frac{e^z dz}{[-\varphi'(x)]^2(1-z^2)}=\int\frac{e^z dz}{[\varphi'(x)]^2(1-z^2)}\quad\text{.................. (1.2.1)}$$

Sub-case I: When φ(x) is linear, let φ(x)=x+b. Then from (1.2.1) we have

$$\int\frac{e^z dz}{[\varphi'(x)]^2(1-z^2)}=\int\frac{e^z dz}{(1-z^2)},$$

which is nonelementary proved in section 1.1 subcase-I.

Sub-case II: When φ(x) is a polynomial of degree 2. Then from (1.2.1) we have

$$\int\frac{e^z dz}{[\varphi'(x)]^2(1-z^2)}=\frac{1}{4}\int\frac{e^z dz}{(\cos^{-1}z+k)(1-z^2)}$$

$$=\int F[z,e^z,\sqrt{1-z^2},\cos^{-1}z]dz$$

A similar argument will hold as in section 1.1 to prove it nonelementary.

1.3. For tangent function, we have on putting $\tan\varphi(x)=z$.

$$\int \frac{e^{f(x)}dx}{f'(x)} = \int \frac{e^{\tan\varphi(x)}dx}{\varphi'(x)\sec^2\varphi(x)} = \int \frac{e^z dz}{[\varphi'(x)]^2(1+z^2)} \qquad \text{.................. (1.3.1)}$$

Sub-case I: When $\varphi(x)$ is linear, let $\varphi(x)=x+b$. Then from (1.3.1) we have

$$\int \frac{e^z dz}{[\varphi'(x)]^2(1+z^2)} = \int \frac{e^z dz}{(1+z^2)} = \frac{1}{2}\left[\int \frac{e^z dz}{(1+iz)} + \int \frac{e^z dz}{(1-iz)}\right]$$

Now

$$\int \frac{e^z dz}{(1+iz)} = \frac{e^i}{i}\int \frac{e^{-ip}}{p}dp, \text{putting } (1+iz)=p$$

By strong Liouville theorem (special case), it is elementary if and only if there exists a rational function R(x) such that it satisfies the identity

$$\frac{1}{p} = R'(p) - iR(p) \Rightarrow R(p) = 0 \text{ and } R'(p) = \frac{1}{p}$$

But R(p) cannot be zero, so such R(p) does not exist. Hence it is nonelementary.

Also

$$\int \frac{e^z dz}{(1-iz)} = ie^{-i}\int \frac{e^{ip}}{p}dp, \text{putting } (1-iz)=p$$

Again by strong Liouville theorem (special case), it is elementary if and only if there exists a rational function R(x) which satisfies the identity

$$\frac{1}{p} = R'(p) + iR(p) \Rightarrow R(p) = 0 \text{ and } R'(p) = \frac{1}{p}$$

But R(p) cannot be zero, so such R(p) does not exist. Hence it is nonelementary. Therefore the given function is nonelementary in this case.

Sub-case II: When $\varphi(x)=x^2+bx+c$. Then from (1.3.1) we have

$$\int \frac{e^z dz}{[\varphi'(x)]^2(1+z^2)} = \frac{1}{4}\int \frac{e^z dz}{[\tan^{-1}z+k](1+z^2)}, k = \frac{b^2-4c}{4}$$

$$= \int F[z, e^z, (1+z^2), \tan^{-1}z]dz = \int F[z, y_1, y_2, y_3]dz$$

$$\left[\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = 2z, \frac{dy_3}{dz} = \frac{1}{1+z^2} = \frac{1}{y_2}\right]$$

By strong Liouville theorem part (b), it is elementary if and only if there exists an identity of the form containing U_j, a function of z, y_1, y_2, and y_3

$$\frac{e^z}{4[\tan^{-1}z+k](1+z^2)}=\frac{dU_i}{dz}+\sum_{i=1}^{n}c_i\frac{U_i'}{U_i}$$

Considering different forms of U_j like $e^z\log(\tan^{-1}z+k)$, $\log[e^z(\tan^{-1}z+k)]$, etc. we find that no such U_j exist, i. e., no such identity exist. Hence the given function is nonelementary. Similarly we can prove it nonelementary for higher degree polynomials φ(x).

1.4. For cotangent function, we have on putting cotφ(x)=z

$$\int\frac{e^{f(x)}dx}{f'(x)}=\int\frac{e^{\cot\varphi(x)}dx}{-\varphi'(x)\operatorname{cosec}^2\varphi(x)}=\int\frac{e^z dz}{[\varphi'(x)]^2(1+z^2)} \quad\text{.................. (1.4.1)}$$

Sub-case-I: For φ(x)=x+b, we have from (1.4.1)

$$\int\frac{e^z dz}{[\varphi'(x)]^2(1+z^2)}=\int\frac{e^z dz}{(1+z^2)}$$

Which is nonelementary, proved in section 1.3, sub-case-I.

Sub-case-II: For $\varphi(x)=x^2+bx+c$, we have from (1.4.1)

$$\int\frac{e^z dz}{[\varphi'(x)]^2(1+z^2)}=\frac{1}{4}\int\frac{e^z dz}{(\cot^{-1}z+k)(1+z^2)},\ k=\frac{b^2-4c}{4}$$

$$=\int F[z,e^z,(1+z^2),\cot^{-1}z]dz$$

A similar argument will hold as in section 1.3 to prove it nonelementary.

1.5. For cosecant function, we have on putting cosecφ(x)=z

$$\int\frac{e^{f(x)}dx}{f'(x)}=\int\frac{e^{\operatorname{cosec}\varphi(x)}dx}{-\varphi'(x)\operatorname{cosec}\varphi(x)\cot\varphi(x)}=\int\frac{e^z dz}{[\varphi'(x)]^2 z^2(z^2-1)} \quad\text{.................. (1.5.1)}$$

Sub-case I: When φ(x) is linear, let φ(x)=x+b. Then from (1.5.1) we have

$$\int\frac{e^z dz}{[\varphi'(x)]^2 z^2(z^2-1)}=\int\frac{e^z dz}{z^2(z^2-1)}=\left[\int\frac{e^z dz}{(z^2-1)}-\int\frac{e^z dz}{z^2}\right]$$

Where the first integral

$$\int \frac{e^z dz}{(z^2-1)}$$

is nonelementary as proved in section 1.1, sub-case-I and the second integral

$$\int \frac{e^z dz}{z^2} = \int z^{-2} e^z dz$$

is also nonelementary from example-5 due to Marchisotto et al [5, pp.301].

Sub-case II: When $\varphi(x)=x^2+bx+c$, then from (1.5.1) we have

$$\int \frac{e^z dz}{[\varphi'(x)]^2 z^2 (z^2-1)} = \int \frac{e^z dz}{[2x+b]^2 z^2 (z^2-1)}$$

$$= \int \frac{e^z dz}{4[\operatorname{cosec}^{-1} z + k] z^2 (z^2-1)}, k = \frac{b^2-4c}{4}$$

$$= \int F[z, e^z, \sqrt{z^2-1}, \operatorname{cosec}^{-1} z] dz = \int F[z, y_1, y_2, y_3] dz$$

$$\left[\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = \frac{z}{\sqrt{z^2-1}} = \frac{z}{y_2}, \frac{dy_3}{dz} = \frac{-1}{|z|\sqrt{z^2-1}} = \frac{-1}{|z| y_2} \right]$$

By strong Liouville theorem part (b), this is elementary if and only if there exists an identity of the form containg U_j, a function of z, y_1, y_2, and y_3 as follows

$$\frac{e^z}{4[\operatorname{cosec}^{-1} z + k] z^2 (z^2-1)} = \frac{dU_i}{dz} + \sum_{i=1}^{n} c_i \frac{U_i'}{U_i}$$

Considering different forms of U_j like $e^z \log[\operatorname{cosec}^{-1}z+k]$, $\log[e^z(\operatorname{cosec}^{-1}z+k)]$, etc., we find that no such U_j exist, which satisfy the above identity. Hence the given function is nonelementary. Similarly we can prove it for higher degree polynomial $\varphi(x)$.

1.6. For secant function, we have on putting $\sec\varphi(x)=z$

$$\int \frac{e^{f(x)} dx}{f'(x)} = \int \frac{e^{\sec\varphi(x)} dx}{\varphi'(x) \sec\varphi(x) \tan\varphi(x)} = \int \frac{e^z dz}{[\varphi'(x)]^2 z^2 (z^2-1)} \quad \text{......................} (1.6.1)$$

Sub-case-I: For $\varphi(x)=x+b$, we have from (1.6.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2 z^2 (z^2-1)} = \int \frac{e^z dz}{z^2 (z^2-1)}$$

Which is nonelementary, proved in section 1.5, sub-case-I.

Sub-case-II: For φ(x)=x^2+bx+c, we have from (1.6.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2 z^2(z^2-1)} = \frac{1}{4}\int \frac{e^z dz}{[\sec^{-1} z + k]z^2(z^2-1)}, k = \frac{b^2-4c}{4}$$

$$= \int F[z, e^z, \sqrt{z^2-1}, \sec^{-1} z]dz$$

It can be proved nonelementary by the similar procedure as has been done in section 1.5.

Case III: **When f(x) be a hyperbolic (not inverse hyperbolic) function.** Let us consider them one by one.

1.7. For sine hyperbolic function, we have on putting sinhφ(x)=z

$$\int \frac{e^{f(x)}dx}{f'(x)} = \int \frac{e^{\sinh\varphi(x)}dx}{\varphi'(x)\cosh\varphi(x)} = \int \frac{e^z dz}{[\varphi'(x)]^2(1+z^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.7.1)$$

Sub-case I: When φ(x) is linear, let φ(x)=x+b. Then from (1.7.1) we have

$$\int \frac{e^z dz}{[\varphi'(x)]^2(1+z^2)} = \int \frac{e^z dz}{(1+z^2)}$$

which is nonelementary, proved in section 1.3, sub-case-I.

Sub-case II: When φ(x)=x^2+bx+c. Then from (1.7.1), we have

$$\int \frac{e^z dz}{[\varphi'(x)]^2(1+z^2)} = \int \frac{e^z dz}{[2x+b]^2(1+z^2)}$$

$$= \int \frac{e^z dz}{4(\sinh^{-1} z + k)(1+z^2)}, \text{where } k = \frac{b^2-4c}{4}$$

$$= \int F[z, e^z, \sqrt{1+z^2}, \sinh^{-1} z]dz = \int F[z, y_1, y_2, y_3]dz$$

$$\left[\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = \frac{z}{\sqrt{1+z^2}} = \frac{z}{y_2}, \frac{dy_3}{dz} = \frac{1}{\sqrt{1+z^2}} = \frac{1}{y_2}\right]$$

Applying strong Liouville theorem part (b), it is elementary if and only if there exists an identity of the form

$$\frac{e^z}{4(\sinh^{-1} z + k)(1+z^2)} = \frac{dU_0}{dz} + \sum_{i=1}^{n} c_i \frac{U'_i}{U_i}.$$

Considering different possible forms of U_j like $e^z\log[\sinh^{-1}z+k]$, $\log[e^z(\sinh^{-1}z+k)]$, etc. we find that no such U_j exist.

Hence the given function is nonelementary. Similarly we can prove it nonelementary for higher degree polynomials.

1.8. For cosine hyperbolic function, we have on putting coshφ(x)=z

$$\int \frac{e^{f(x)}dx}{f'(x)} = \int \frac{e^{\cosh\varphi(x)}dx}{\varphi'(x)\sinh\varphi(x)} = \int \frac{e^z dz}{[\varphi'(x)]^2(z^2-1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1.8.1)$$

Sub-case-I: For φ(x)=x+b, we have from (1.8.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2(z^2-1)} = \int \frac{e^z dz}{(z^2-1)}$$

Which is nonelementary proved in section 1.1, sub-case-I.

Sub-case-II: For $\varphi(x)=x^2+bx+c$, we have from (1.8.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2(z^2-1)} = \frac{1}{4}\int \frac{e^z dz}{(\cosh^{-1}z+k)(z^2-1)}, k = \frac{b^2-4c}{4}$$

$$= \int F[z, e^z, \sqrt{z^2-1}, \cosh^{-1}z]dz$$

It can now be proved nonelementary by strong Liouville theorem part (b). Similarly we can prove it for higher degree polynomial φ(x).

1.9. For tangent hyperbolic function, we have on putting tanhφ(x)=z

$$\int \frac{e^{f(x)}dx}{f'(x)} = \int \frac{e^{\tanh\varphi(x)}dx}{\varphi'(x)\operatorname{sech}^2\varphi(x)} = \int \frac{e^z dz}{[\varphi'(x)]^2(1-z^2)} \quad \ldots\ldots\ldots\ldots\ldots\ (1.9.1)$$

Sub-case-I: For φ(x)=x+b, we have from (1.9.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2(1-z^2)} = \int \frac{e^z dz}{(1-z^2)}$$

Which is nonelementary, proved in section 1.1, sub-case-I.

Sub-case-II: For $\varphi(x)=x^2+bx+c$, we have from (1.9.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2(1-z^2)} = \frac{1}{4}\int \frac{e^z dz}{(\tanh^{-1}z+k)(1-z^2)}, k = \frac{b^2-4c}{4}$$

$$= \int F[z, e^z, (1-z^2), \tanh^{-1}z]dz$$

It can now be proved nonelementary by strong Liouville theorem part (b). Similarly we can prove it for higher degree polynomial φ(x).

1.10. For cotangent hyperbolic function, we have on putting cothφ(x)=z

$$\int \frac{e^{f(x)}dx}{f'(x)} = \int \frac{e^{\cot\varphi h(x)}dx}{-\varphi'(x)\operatorname{cosech}^2\varphi(x)} = \int \frac{e^z dz}{[\varphi'(x)]^2(z^2-1)} \quad \ldots\ldots\ldots\ldots (1.10.1)$$

Sub-case-I: For φ(x)=x+b, we have from (1.10.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2(z^2-1)} = \int \frac{e^z dz}{(z^2-1)}$$

Which is nonelementary proved in section 1.1, sub-case-I.

Sub-case-II: For φ(x)=x^2+bx+c, we have from (1.10.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2(z^2-1)} = \frac{1}{4}\int \frac{e^z dz}{(\coth^{-1}z+k)(z^2-1)}, k = \frac{b^2-4c}{4}$$

$$= \int F[z, e^z, (z^2-1), \coth^{-1}z]dz$$

It can now be proved nonelementary by strong Liouville theorem part (b). Similarly we can prove it for higher degree polynomial φ(x).

1.11. For cosecant hyperbolic function, we have on putting cosechφ(x)=z

$$\int \frac{e^{f(x)}dx}{f'(x)} = \int \frac{e^{\operatorname{cosec}\varphi h(x)}dx}{-\varphi'(x)\operatorname{cosech}\varphi(x)\coth\varphi(x)} = \int \frac{e^z dz}{[\varphi'(x)]^2 z^2(z^2+1)} \quad \ldots\ldots\ldots\ldots\ldots (1.11.1)$$

Sub-case-I: For φ(x)=x+b, we have from (1.11.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2 z^2(z^2+1)} = \int \frac{e^z dz}{z^2(z^2+1)} = \int \frac{e^z dz}{z^2} - \int \frac{e^z dz}{z^2+1}$$

Both are nonelementary proved in section 1.5, sub-case-I and section 1.7, sub-case-I respectively.

Sub-case-II: For φ(x)=x^2+bx+c, we have from (1.11.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2 z^2(z^2+1)} = \frac{1}{4}\int \frac{e^z dz}{(\operatorname{cosech}^{-1}z+k)z^2(z^2+1)}, k = \frac{b^2-4c}{4}$$

$$= \int F[z, e^z, \sqrt{z^2+1}, \operatorname{cosech}^{-1}z]dz$$

It can now be proved nonelementary by strong Liouville theorem part (b). Similarly we can prove it for higher degree polynomial φ(x).

1.12. For secant hyperbolic function, we have on putting sechφ(x)=z

$$\int \frac{e^{f(x)}dx}{f'(x)} = \int \frac{e^{\mathrm{sech}\,\varphi(x)}dx}{-\varphi'(x)\,\mathrm{sech}\,\varphi(x)\tanh\varphi(x)} = \int \frac{e^z dz}{[\varphi'(x)]^2 z^2(1-z^2)} \qquad \text{……………} \; (1.12.1)$$

Sub-case-I: For $\varphi(x)=x+b$, we have from (1.12.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2 z^2(1-z^2)} = \int \frac{e^z dz}{z^2(1-z^2)} = \int \frac{e^z dz}{z^2} + \int \frac{e^z dz}{1-z^2}$$

Both are nonelementary proved in section 1.5, sub-case-I and section 1.1, sub-case-I respectively.

Sub-case-II: For $\varphi(x)=x^2+bx+c$, we have from (1.12.1)

$$\int \frac{e^z dz}{[\varphi'(x)]^2 z^2(1-z^2)} = \frac{1}{4}\int \frac{e^z dz}{(\mathrm{sech}^{-1}z+k)z^2(1-z^2)}$$

$$= \int F[z, e^z, \sqrt{1-z^2}, \mathrm{sech}^{-1} z]dz$$

It can now be proved nonelementary by strong Liouville theorem part (b). Similarly we can prove it for higher degree polynomial $\varphi(x)$.

Let us consider some examples on this standard form of nonelementary functions:

Example 1: Show that the integral $\int \frac{e^{ax^2+b}}{x}dx,\; a \neq 0$ is nonelementary.

Proof: We have

$$\int \frac{e^{ax^2+b}}{x}dx = \int \frac{e^{ax^2}}{x}dx + \int \frac{e^b}{x}dx = e^b \log x + \int \frac{2axe^{ax^2}}{2ax^2}dx$$

Now for second integral, putting $ax^2=z$ we have

$$\int \frac{2axe^{ax^2}}{2ax^2}dx = \frac{1}{2}\int \frac{e^z}{z}dz = \frac{1}{2}\int z^{-1}e^z dz$$

which is nonelementary from example-5 due to Marchisotto et al [5, pp.301].

Example 2: Show that the integral $\int \frac{e^{\sin x}}{\cos x}dx$ is nonelementary.

Proof: We have $\int \frac{e^{\sin x}}{\cos x}dx = \int \frac{e^{\sin x}\cos x}{\cos^2 x}dx = \int \frac{e^z dz}{(1-z^2)}$

On putting sinx=z. Which is nonelementary, proved in section 1.1, sub-case-I.

Example 3: Show that the integral $\int \frac{e^{\cos x}}{-\sin x}dx$ is nonelementary.

Proof: We have

$$\int \frac{e^{\cos x}}{-\sin x} dx = \int \frac{e^{\cos x}(-\sin x)}{(-\sin x)^2} dx = \int \frac{e^z dz}{(1-z^2)}$$

On putting cosx=z. Which is nonelementary proved in section 1.1, sub-case-I.

Example 4: Show that the integral $\int \frac{e^{\tan x}}{\sec^2 x} dx$ is nonelementary.

Proof: We have, on putting tanx=z

$$\int \frac{e^{\tan x}}{\sec^2 x} dx = \int \frac{e^z}{(1+z^2)^2} dz = \frac{1}{4}\int \frac{e^z dz}{(iz)(1-iz)^2} - \frac{1}{4}\int \frac{e^z dz}{(iz)(1+iz)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

We have on putting 1-iz = p in the first integral of (A)

$$\int \frac{e^z dz}{(iz)(1-iz)^2} = ie^{-i}\left[\int \frac{e^{ip} dp}{(1-p)} + \int \frac{e^{ip} dp}{p} + \int \frac{e^{ip} dp}{p^2}\right] \quad \ldots\ldots\ldots\ldots \text{(B)}$$

where the second and third integrals are nonelementary from example-5 due to Marchisotto et al [5, pp.301]. Now putting 1-p=X in the first integral of (B) we have

$$\int \frac{e^{ip} dp}{(1-p)} = -e^{i}\int \frac{e^{-iX} dX}{X}$$

which is also nonelementary from example-5 due to Marchisotto et al [5, pp.301]. Therefore the first integral of (A) is nonelementary. Similarly we can prove that the second integral of (A) is also nonelementary. Therefore the given function is nonelementary.

Example 5: Show that the integral $\int \frac{e^{\sinh x}}{\cosh x} dx$ is nonelementary.

Proof: We have on putting sinhx=z

$$\int \frac{e^{\sinh x}}{\cosh x} dx = \int \frac{e^z}{(1+z^2)} dz$$

Which is nonelementary proved in section 1.3, sub-case-I.

Example 6: Show that the integral $\int \frac{e^{\cot x}}{-\operatorname{cosec}^2 x} dx$ is nonelementary.

Proof: We have on putting z=cotx

$$\int \frac{e^{\cot x}}{-\operatorname{cosec}^2 x} dx = \int \frac{e^z}{(1+z^2)} dz$$

Which is nonelementary proved in section 1.3, sub-case-I.

Example 7: Show that the integral $\int \frac{e^{\sec x}}{\sec x.\tan x}dx$ is nonelementary.

Proof: We have on putting secx=z

$$\int \frac{e^{\sec x}}{\sec x.\tan x}dx = \int \frac{e^z}{z^2(z^2-1)}dz = \int \frac{e^z}{(z^2-1)}dz - \int \frac{e^z}{z^2}dz$$

Which are nonelementary, proved in section 1.5, sub-case-I.

Example 8: Show that the integral $\int \frac{e^{\operatorname{cosec} x}}{-\operatorname{cosec} x.\cot x}dx$ is nonelementary.

Proof: We have on putting cosecx=z,

$$\int \frac{e^{\operatorname{cosec} x}}{-\operatorname{cosec} x.\cot x}dx = \int \frac{e^z}{z^2(z^2-1)}dz = \int \frac{e^z}{(z^2-1)}dz - \int \frac{e^z}{z^2}dz$$

Which is nonelementary, proved in section 1.5, sub-case-I.

Example 9: Show that the integral $\int \frac{e^{\sin^2 x}}{\sin 2x}dx$ is nonelementary.

Proof: We have on putting $\sin^2$x=z,

$$\int \frac{e^{\sin^2 x}}{\sin 2x}dx = \frac{1}{4}\int \frac{e^z dz}{z(1-z^2)} = \frac{1}{4}\left[\int \frac{ze^z dz}{(1-z^2)} + \int \frac{e^z dz}{z}\right]$$

Where $\int \frac{e^z}{z}dz$ is nonelementary from example-5 due to Marchisotto et al [5, pp.301].

Now since $I = \int \frac{ze^z}{(1-z^2)}dz = \frac{1}{2}\int \frac{e^z dz}{(1-z)} - \frac{1}{2}\int \frac{e^z dz}{(1+z)}$

Where $\int \frac{e^z dz}{(1-z)} = -e\int \frac{e^{-p}}{p}dp$, on putting 1-z=p, which is nonelementary

and

$\int \frac{e^z dz}{(1+z)} = \frac{1}{e}\int \frac{e^p}{p}dp$ on putting 1+z=p, which is also nonelementary

from example-5 due to Marchisotto et al [5, pp.301]. Hence the given function is nonelementary.

Example 10: Show that the integral $\int \frac{e^{\sin x^2}}{2x.\cos x^2}dx$ is nonelementary.

Proof: We have on putting $\sin x^2 = z$,

$$\int \frac{e^{\sin x^2}}{2x.\cos x^2}dx = \int \frac{e^z dz}{4(1-z^2)\sin^{-1} z}$$

$$= \int F\left(z, e^z, \sqrt{1-z^2}, \sin^{-1} z\right)dz = \int F(z, y_1, y_2, y_3)dz$$

$$\left[\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = \frac{-z}{\sqrt{1-z^2}} = \frac{-z}{y_2}, \frac{dy_3}{dz} = \frac{1}{\sqrt{1-z^2}} = \frac{1}{y_2}\right]$$

Applying strong Liouville theorem, part(b), it is elementary if and only if there exists an identity of the form, containing U_i a function of z, y_1, y_2, and y_3 as

$$\frac{dU_o}{dz} + \sum_{i=1}^{n} C_i \frac{U'_i}{U_i} = \frac{e^z}{(1-z^2)\sin^{-1} z}$$

Taking different possible forms of U_j we find that no such U_j exist. Hence the given function is nonelementary.

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