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A NECESSARY AND SUFFICIENT CONDITION FOR A SIGNED GRAPH TO BE A REGULAR SIGNED GRAPH

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ABSTRACT

A signed graph is a graph S = (G,σ), where G = (V,E) in which each edge is assigned a positive or negative sign. Let S = (G,σ) be a signed graph with V = $(v_1, v_2, \ldots, v_n)$. In this paper we obtain the necessary and sufficient condition for a signed graph to be a regular signed graph.

Keywords and Phrases.- Simple Graph, Multi-Graph, Signed Graph, Signed Degree.

I.INTRODUCTION

A signed graph is defined to be a pair S = (G,σ), where G = (V,E) is the underlying graph and σ : E → {−1,1} is the signing function. These were first introduced by Harary [3]. The sets of positive and negative edges of S are respectively denoted by E^+ and E^-. Thus E =$E^+ \cup E^-$. Our signed graphs have simple underlying graphs. A signed graph is said to be homogeneous if all of its edges have either positive sign or negative sign and heterogeneous, otherwise. A graph can be considered to be a homogeneous signed graph with each edge positive; thus signed graphs become a generalization of graphs. The sign of a signed graph is defined as the product of signs of its edges. A signed graph is said to be positive (respectively, negative) if its sign is positive (respectively, negative) i.e., it contains an even (respectively, odd) number of negative edges. A signed graph is said to be all-positive (respectively, all-negative) if all of its edges are positive (respectively, negative). A signed graph is said to be balanced if each of its cycles is positive and unbalanced, otherwise. We denote by −S the signed graph obtained by negating each edge of S and call it the negative of S. We call balanced cycle a positive cycle and an unbalanced cycle a negative cycle and respectively denote them by Cn and C_n, where n is number of vertices.

The signed degree of v_i is sdeg(v_i) = $d_i = {d_i}^+ + {d_i}^-$, where 1 ≤ i ≤ n and ${d_i}^+$ (${d_i}^-$) is the number of positive(negative) edges incident with v_i. So the sequence σ = $(d_1, d_2, \ldots, d_n)$ in non-increasing order is called signed degree sequence of G. We denote poitive edge xy by xy+ and a negative edge xy by xy−. An integral sequence is s-graphical if it is the signed degree sequence of a signed graph. Also, a non-zero sequence

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σ = $(d_1, d_2,..., d_n)$ is a standard sequence if σ is non-increasing, $\sum_{i=1}^{n} d_i$di is even, d_1 >0, each $|d_i|$<n and $|d_i| \geq |d_n|$.

Definition 1.1.

The signed star graph denoted by $K_{1,n}$ is is the signed bipartite graph with partite sets 1 and n. The following result, due to Chartrand et al.[1], gives a necessary and sufficient condition for an integral sequence to be sgraphical, which is similar to Hakimi's result for graphical sequences in graphs [2].

Theorem 1.2. A standard integral sequence σ = $(d_1, d_2,..., d_n)$ is s-graphical if and only if $\sigma^{/}$ = $(d_2-1, d_3-1, \cdots, d_{d_{1+s+1}}, d_{d_{1+s+2}}, d_{n-s}, d_{n-s+1}+1, \cdots, d_n+1)$ is s-graphical for some s, $0 \leq s \leq n-1-d_1$.

The next characterization for signed graphical sequences in signed graphs is given by Yan et al. [7]

Theorem 1.3. A standard integral sequence σ = $(d_1, d_2,..., d_n)$ is s-graphical if and only if $\sigma_m{}^{/}$ = $(d_2-1, d_3-1, \cdots, d_{d_{1+s+1}}, d_{d_{1+s+2}}, d_{n-m}, d_{n-m+1}+1, \cdots, d_n+1)$ is s-graphical where m is the maximum non-negative integer such that , $d_{d_{1+m+1}} > d_{n-m+1}$.

The set of distinct signed degrees of the vertices in a signed graph is called its signed degree set. Pirzada et al. [4] proved that every non-empty set of positive(negative) integers is the signed degree set of some connected signed graph and determine the smallest possible order for such a signed graph. In 2015, Pirzada et al. [6] obtained graphical sequences of some family of induced subgraphs and obtained new condition for a graphic sequence to be potentially K4 −e graphic. A complete signed bipartite graph with r vertices in one partite set and s vertices in another set is denoted by Kr,s is a complete bipartite graph in which each edge is assigned a positive or a negative sign. A complete signed bipartite graph of the form K1,n−1 is called a star signed graph.
In 2007 Pirzada et al. [5] proved the following assertions in signed bipartite graphs.

Theorem 1.4. Let S(U,V) be a signed bipartite graph with m edges. Then, g $=\sum_{u\in U}^{n} sdeg_{(u)}$ = $\sum_{v\in V}^{n} sdeg_{(v)}$ sdeg(u) ≡ m(mod2) and the number of positive edges and negative edges of G(U,V) are respectively m+g/ 2 and m−g/ 2 .

The following result gives a necessary and sufficient condition for a pair of integral sequences to be the signed degree sequences of some complete signed bipartite graph.

Theorem 1.5. Let α= $(d_1, d_2,..., d_p)$and β = $(e_1, e_2,..., e_q)$be standard sequences and let r = $d_1 + q_2$.

Let α0 be obtained from α by deleting d_1 and β0 be obtained from β by reducing r greatest entries of β by 1 each and adding remaining entries of β by 1 each. Then α and β are the signed graphic sequences of some complete signed bipartite graph if and only if α0 and β0 are also.

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II MAIN RESULTS

The following result gives the necessary and sufficient condition for a signed graph to be a regular signed graph.

Theorem 2.1. A Signed graph S = (V,ζ) is regular if and only if $|\zeta|^2 = n/4\sum_{i=1}^{n} |(sdeg v_i)|^2$.

Proof. Suppose that the signed graph S = (V,ζ) is a regular of degree r. Therefore 2|ζ| = nr and $sdeg v_i = r$ ∀ i = 1, 2, ⋯ ,n.

We know that $(sdeg v_1)^2 + (sdeg v_2)^2 + \cdots + (sdeg v_n)^2 = ({d_1}^+ - {d_1}^-)^2 + ({d_2}^+ - {d_2}^-)^2 + \cdots + ({d_n}^+ - {d_n}^-)^2 = n\,({d_1}^+ - {d_1}^-)^2 = nr^2$.

Further, we have

$$4\,|\zeta|^2/n = 4/n\,.\,n^2\,r^2/4 = n\,r^2 \qquad (1)$$

Therefore, from above we have

$$(sdeg v_1)^2 + (sdeg v_2)^2 + \cdots + (sdeg v_n)^2 = 4\,|\zeta|^2/n$$

$$\Rightarrow |\zeta|^2 = n/4\sum_{i=1}^{n} (sdeg v_i)^2 .$$

Conversely, suppose that $n/4\sum_{i=1}^{n} (sdeg v_i)^2 = |\zeta|^2$.

$$\Rightarrow n/4\{((sdeg v_1)^2 + (sdeg v_2)^2 + \cdots + (sdeg v_n)^2\} - |\zeta|^2 = 0$$

$$\Rightarrow (sdeg v_1)^2 + (sdeg v_2)^2 + \cdots + (sdeg v_n)^2 - 1/n\,|2\zeta|^2 = 0$$

$$\Rightarrow (sdeg v_1)^2 + (sdeg v_2)^2 + \cdots + (sdeg v_n)^2 - 1/n\{(sdeg v_1)^2 + (sdeg v_2)^2$$
$$+ \cdots + (sdeg v_n)^2 + 2(sdeg v_1\, sdeg v_2 + sdeg v_1\, sdeg v_3 + \cdots + sdeg v_1\, sdeg v_n)$$
$$+ \quad 2(sdeg v_2\, sdeg v_3 + sdeg v_2\, sdeg v_4 + \cdots + sdeg v_2\, sdeg v_n)$$
$$+ \quad 2\, sdeg v_{n-1} - 1\, sdeg v_n = 0$$

$$\Rightarrow 1/n\ (sdeg v_1 - sdeg v_2)^2 + \cdots + (sdeg v_1 - sdeg v_n)^2 +$$
$$(sdeg v_2 - sdeg v_3)^2 + \cdots + (sdeg v_2 - sdeg v_n)^2 + \cdots +$$
$$(sdeg v_{n-1} - sdeg v_n)^2 = 0.$$

From this equation each term is non-negative for every i, j. Therefore, we know that it is possible only when $sdeg v_i = sdeg v_j$ for every i, j and therefore signed graph S = (V,ζ) is a regular signed graph. Hence completes the proof.

The following result gives the condition for a signed graph to be a complete signed graph.

Theorem 2.2. Let S be a signed graph with n ≥ 2 vertices. Then S is a complete signed graph Kn if and only if

$$|\zeta|^2 = n/4\sum_{i=1}^{n} (sdeg v_i)^2 = |\zeta|^2\{2\,\zeta/n-1 + n-2\}$$

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Proof. We know that a signed graph S is complete iff $|\zeta| = n(n-1)/2$

$$\Leftrightarrow \quad 2|\zeta| = n(n-1)$$

$$\Leftrightarrow \quad 2|\zeta|(n-2) = n(n-1)(n-2)$$

$$\Leftrightarrow \quad 2|\zeta|(n-2) + 2|\xi|n = 2|\zeta|n + n(n-1)(n-2)$$

$$\Leftrightarrow \quad 4\,|\zeta|^2/n = |\zeta|^2\{2\,\zeta/n-1 + n-2\}$$

Thus we have seen that a signed graph $S(V,\zeta)$ is complete if and only if

$$4\,|\zeta|^2/n = |\zeta|^2\{2\,\zeta/n-1 + n-2\}$$

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