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Statistical $(\overline{N}, p_n, q_n)(E, q)$ summablility and its approximation theorems

Tejaswini Pradhan[1*], Priyadarshini Parida[2†]and Susanta Kumar Paikray[3‡]
[1,3]Department of Mathematics, VSS University of Technology, Burla - 768018, Odisha, India
[2]Research Scholar, Department of Mathematics, Ravenshaw University, Cuttack 753003, Odisha, India

ABSTRACT

Statistical convergence has attracted the attention of the current researchers due basically to the fact that, it is stronger than classical convergence. Korovkin type approximation theory plays an important role for convergence analysis for sequences of positive operators. Further, the approximation theorem has been extended to more general space of sequences via different statistical summability methods. In this proposed paper, we have introduced presumably a new statistical product $(\overline{N}, p_n, q_n)(E, q)$ summability mean to prove a Korovkin type approximation theorem. Furthermore, we have established a new result for the rate of convergence which is uniform in Korovkin type theorem under our defined summability mean.

Key words: Korovkin theorem; positive linear operator; statistical convergence; (E, q) mean; $(\overline{N}, p_n, q_n)$ mean;

AMS Subject classification (2010): 41A24; 41A25; 42B05; 42B08

I. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

The concept of statistical convergence for real sequences was first introduced by Fast [6] in 1951. But previously the idea of statistical convergence was given by Zygmund [21] in 1935. Using real and complex sequences it was further investigated by Schoenberg [19] independently. Continuously several researchers are doing research in this statistical convergence and it plays important role in different areas of mathematics such as Fourier analysis, ergodic theory, approximation theory, Number theory and Functional analysis. Later on, statistical convergence was investigated from the sequence point of view and linked with summability theory by Conner [4] and Fridy [7] and became a versatile field of research in last decades. Moricz [13] introduced statistical $(C, 1)$ summability. Through statistical $(C, 1)$ summability Mohiuddine *et.al.* [16], Mohiuddine [14] and Mohiuddine and Alotaibi [15] proved Korovkin type approximation theorem by using different test function $1, \sin x, \cos x$ and $1, e^{-x}, e^{-2x}$. Belen *et.al.* [2] proved approximation theorems by generalized statistical convergence. Subsequently, many researchers obtained korovkin's type approximation theorem through generalized statistical convergence, generalized equi-statistical convergence, A-statistical convergence etc. using different summation processes are seen in [5], [3], [18], [20]. Recently Acar and Mohiuddine [1] introduced statistical $(C, 1)(E, 1)$ summability and its applications to Korovkin's theory. Using the concept of statistical $(C, 1)(E, 1)$ summability in this paper we have introduced a new statistical $(\overline{N}, p_n, q_n)(E, q)$ summablity using product of $(\overline{N}, p_n, q_n)$ and (E, q) mean. Also we investigate the rate of statistical $(\overline{N}, p_n, q_n)(E, q)$ summablility and obtain a Korovkin-type approximation theorem for positive linear operator by means of statistical $(\overline{N}, p_n, q_n)(E, q)$ summability.

Here we begin with some basic defnitions concerning statistical convergence. Let $\mathbb{N}$ be the set of positive integers and let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined as

$$\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n},$$

if the limit exists.
A sequence (u_n) of real numbers is said to be statistically convergent to 'l' if, for every $\epsilon \geq 0$, the set

$$\{n \in \mathbb{N} : |u_n - l| \geq \epsilon\}$$

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has natural density zero. i.e., for each $\epsilon \geq 0$, we have

$$\delta(\{n \in \mathbb{N} : |u_n - l| \geq \epsilon\}) = 0.$$

In this case, we write this as

$$st - \lim_{n\to\infty} u_n = l.$$

Moreover, statistical convergence is more general than the classical convergence, that is a sequence can be convergent in statistical mean even if it isn't convergent in claasical mean. Also it has been seen that statistical convergence is closely related to the concept of convergence in probability.

Let define the statistical product $(\overline{N}, p_n, q_n)(E, q)$ summability mean. Suppose (p_n) and (q_n) be two non negetive real sequences such that

$$P_n = p_1 + p_2 + ... + p_n, \quad P_{-1} = p_{-1} = 0,$$

$$Q_n = q_1 + q_2 + ... + q_n, \quad Q_{-1} = q_{-1} = 0.$$

Product of above two sequences will be defined as

$$R_n = \sum_{k=0}^{n} p_k q_k.$$

If the sequence to sequence transformation

$$t_n^{\overline{N}} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k u_k,$$

converges to 'l' as $n \to \infty$, then the sequence (u_n) is summable to 'l' by $(\overline{N}, p_n, q_n)$ summability.
The sequence (u_n) is statistically summable to 'l' by $(\overline{N}, p_n, q_n)$ summability method generated by the sequences (p_n) and (q_n) if,

$$st - \lim_{n\to\infty} t_n^{\overline{N}} = l.$$

We also write this as $(\overline{N}_{p_n q_n})(st) - \lim_{n\to\infty} u_n = l$.
If the sequence to sequence transformation

$$E_n^q = \frac{1}{(1+q)^n} \sum_{r=0}^{k} \binom{k}{r} u_r \qquad \text{for } q > 0$$

converges to 'l' as $n \to \infty$, then the sequence (u_n) is summable to 'l' by (E, q) summability.
The sequence (u_n) is statistically summable to 'l' by (E, q) summability if,

$$st - \lim_{n\to\infty} E_n^q = l.$$

We also write this as $(E^q)(st) - \lim_{n\to\infty} u_n = l$.
Now we define a composite transformation, i.e. the $(\overline{N}, p_n, q_n)$ transformation over (E, q) transformation as

$$(\overline{N}_{p_n q_n}.E^q) = T_n^{\overline{N}E} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k \left\{ \frac{1}{(1+q)^n} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} u_r \right\} \tag{1.1}$$

If $T_n^{\overline{N}E} \to l$ as $n \to \infty$, we say (u_n) is summable to 'l' by $(\overline{N}, p_n, q_n)(E, q)$ summability.
The sequence (u_n) is said to be statistically summable to 'l' by $(\overline{N}, p_n, q_n)(E, q)$ summability if,

$$st - \lim_{n\to\infty} T_n^{\overline{N}E} = l. \tag{1.2}$$

We also write this as $(\overline{N}_{p_n q_n}.E^q)(st) - \lim_{n\to\infty} u_n = l$.

Remark 1: If we put $p_n = 1$, $q_n = 1$ and $R_n = n + 1$ in (1.1), then we have

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$$
\begin{aligned}
T_n^{\overline{N}E} &= \frac{1}{R_n}\sum_{k=0}^{n} p_k q_k \left\{ \frac{1}{(1+q)^k}\sum_{r=0}^{k}\binom{k}{r} q^{k-r} u_r \right\} \\
&= \frac{1}{n+1}\sum_{k=0}^{n} \left\{ \frac{1}{(1+q)^k}\sum_{r=0}^{k}\binom{k}{r} q^{k-r} u_r \right\} \qquad (1.3)
\end{aligned}
$$

which is the Cesàro-Euler summation. Hence, if we put $p_n = 1$ and $q_n = 1$ in (1.1), the generalized $(\overline{N}, p_n, q_n)(E, q)$ summability method reduced to $(C, 1)(E, q)$ summability.

Example-1: Let us consider $p_n = 1$ and $q_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Also consider the following sequence

$$u_n = (-2)^n, \ n \in \mathbb{N}$$

This sequence (u_n) is not statistically summable but $(\overline{N}, p_n, q_n)(E, q)$ summable. i.e.
if we put $p_n = 1$, $q_n = \frac{1}{n}$ and $u_n = (-2)^n$ for all $n \in \mathbb{N}$ in (1.1), then the equation becomes

$$\frac{1}{n+1}\sum_{k=0}^{n} 1 \left\{ \frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v} q^{k-v}(-2)^v \right\} = \frac{1}{n+1}\sum_{k=0}^{n}\frac{(q-2)^k}{(q+2)^k} \qquad (1.4)$$

as $n \to \infty$, equation (1.4) converges.

II. KOROVKIN TYPE APPROXIMATION THEOREM

The Korovkin type approximation theorems are investigated by several mathemeticians and they generalize this Korovkin type approximation theorems in many different ways using various test functions in several setups in function spaces, abstract Banach latices, Banach algebras and so on. This concept is very much useful in real analysis, harmonic analysis, functional analysis etc. The classical Korovkin first theorem are given in [1], [8], [11], [12], [18]. In this section we prove a Korovkin type approximation theorem via statistical $(\overline{N}, p_n, q_n)(E, q)$ summability.

Let $\mathbb{C}[a, b]$ be the space of all real-valued continuous functions f defined on $[a, b]$ and $(\mathcal{A})$ be a positive linear operator which maps $\mathbb{C}[a, b]$ into $\mathbb{C}[a, b]$. And this space is equipped with the norm

$$\|g\|_\infty = \sup_{x \in [a,b]} |g(x)|$$

Following [11], the classical Korovkin approximation theorem stated as follows;
Theorem 1: Let (T_n) be a sequence of positive linear operators from $\mathbb{C}[a, b]$ into $\mathbb{C}[a, b]$. Then,

$$\lim_{n\to\infty} \|T_n(f; x) - f(x)\|_\infty = 0 \qquad (2.5)$$

for all $f \in \mathbb{C}[a, b]$ if and only if

$$\lim_{n\to\infty} \|T_n(f_i; x) - f_i(x)\|_\infty = 0 \qquad (2.6)$$

where $f_i(x) = x^i$ and i=0,1,2.

The main objective of this paper is to prove the following theorem.

Theorem 2: Let $(\mathcal{A}_k)$ be a sequence of positive linear operators which maps $\mathbb{C}[a, b]$ into $\mathbb{C}[a, b]$. Then for all $f \in \mathbb{C}[a, b]$ bounded on the whole real line,

$$(\overline{N}_{p_n q_n}.E^q)(st) - \lim_{k\to\infty} \|\mathcal{A}_k(f; x) - f(x)\|_\infty = 0, \qquad (2.7)$$

if and only if

$$(\overline{N}_{p_n q_n}.E^q)(st) - \lim_{k\to\infty} \|\mathcal{A}_k(1; x) - 1\|_\infty = 0, \qquad (2.8)$$

$$(\overline{N}_{p_n q_n}.E^q)(st) - \lim_{k\to\infty} \|\mathcal{A}_k(t; x) - x\|_\infty = 0, \qquad (2.9)$$

$$(\overline{N}_{p_n q_n}.E^q)(st) - \lim_{k\to\infty} \|\mathcal{A}_k(t^2; x) - x^2\|_\infty = 0. \qquad (2.10)$$

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Proof: The equations (2.8) to (2.10) immediately follows from (2.7) because of each $1, x, x^2$ belongs to $\mathbb{C}[a,b]$. Next we prove its converse part. i.e. if the conditions (2.8) to (2.10) holds true, then (2.7) is valid. Let $f \in \mathbb{C}[a,b]$, then there exist a constant $C > 0$ such that

$$|f(x)| \leq C \quad \text{for} \quad \text{all } x \in (-\infty, \infty), \tag{2.11}$$

and therefore

$$|f(t) - f(x)| \leq 2C, \quad -\infty < t, x < \infty. \tag{2.12}$$

We may write, for every $\epsilon > 0$, than there exist a number $\delta > 0$ such that

$$|f(t) - f(x)| \leq \epsilon, \text{for} \quad \text{all } |t - x| \leq \delta \tag{2.13}$$

Combining (2.12) and (2.13) and substituting $\phi(t) = (t - x)^2$, we get

$$|f(t) - f(x)| \leq \epsilon + \frac{2C}{\delta^2}\phi(t) \tag{2.14}$$

Now (2.14) can be written as

$$-\epsilon - \frac{2C}{\delta^2}\phi(t) \leq f(t) - f(x) \leq \epsilon + \frac{2C}{\delta^2}\phi(t). \tag{2.15}$$

As the operators are positive, operating $\mathcal{A}_k(1; x)$ to (2.15), we get

$$\mathcal{A}_k(1; x)\left(-\epsilon - \frac{2C}{\delta^2}\phi(t)\right) \leq \mathcal{A}_k(1; x)(f(t) - f(x)) \leq \mathcal{A}_k(1; x)\left(\epsilon + \frac{2C}{\delta^2}\phi(t)\right) \tag{2.16}$$

We know x is fixed, so $f(x)$ is a constant number and using the linearity property of $(\mathcal{A}_k)$, we have

$$-\epsilon\mathcal{A}_k(1; x) - \frac{2C}{\delta^2}\mathcal{A}_k(\phi(t); x) \leq \mathcal{A}_k(f; x) - f(x)\mathcal{A}_k(1; x) \leq \epsilon\mathcal{A}_k(1; x) + \frac{2C}{\delta^2}\mathcal{A}_k(\phi(t); x) \tag{2.17}$$

The term $\mathcal{A}_k(f; x) - f(x)$ can be writeen as

$$\mathcal{A}_k(f; x) - f(x) = \mathcal{A}_k(f; x) - f(x)\mathcal{A}_k(1; x) + f(x)[\mathcal{A}_k(1; x) - 1] \tag{2.18}$$

Now considering the inequality (2.17) and equality (2.18), we get

$$\mathcal{A}_k(f; x) - f(x) < \epsilon\mathcal{A}_k(1; x) + \frac{2C}{\delta^2}\mathcal{A}_k(\phi(t); x) + f(x)[\mathcal{A}_k(1; x) - 1]. \tag{2.19}$$

The term $\mathcal{A}_k(\phi(t); x)$ can be written as follows:

$$\mathcal{A}_k(\phi(t); x) = \left[\mathcal{A}_k(t^2; x) - x^2\right] - 2x[\mathcal{A}_k(t; x) - x] + x^2[\mathcal{A}_k(1; x) - 1]. \tag{2.20}$$

Now putting the value of $\mathcal{A}_k(\phi(t); x)$ in (2.19), we get

$$\begin{aligned}
\mathcal{A}_k(f; x) - f(x) \quad &< \quad \mathcal{A}_k(1; x) + \frac{2C}{\delta^2}\left\{\left[\mathcal{A}_k(t^2; x) - x^2\right] - 2x[\mathcal{A}_k(t; x) - x] + x^2[\mathcal{A}_k(1; x) - 1]\right\} \\
&\quad + f(x)[\mathcal{A}_k(1; x) - 1] \\
&= \quad \epsilon[\mathcal{A}_k(1; x) - 1] + \epsilon + f(x)[\mathcal{A}_k(1; x) - 1] \\
&\quad + \frac{2C}{\delta^2}\left\{\left[\mathcal{A}_k(t^2; x) - x^2\right] - 2x[\mathcal{A}_k(t; x) - x] + x^2[\mathcal{A}_k(1; x) - 1]\right\}
\end{aligned} \tag{2.21}$$

If we take $h = \max|x|$, then (2.21) becomes

$$\begin{aligned}
|\mathcal{A}_k(f; x) - f(x)| \quad &< \quad \left((\epsilon + \frac{2Ch^2}{\delta^2} + C)|\mathcal{A}_k(1; x) - 1|\right) \\
&\quad + \frac{4Ch}{\delta^2}|\mathcal{A}_k(t; x) - x| \\
&\quad + \frac{2C}{\delta^2}|\mathcal{A}_k(t^2; x) - x^2|
\end{aligned} \tag{2.22}$$

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Now taking suprimum over all $x \in [a,b]$, we have

$$\begin{aligned} \|\mathcal{A}_k(f;x) - f(x)\|_\infty &< \left((\epsilon + \frac{2Ch^2}{\delta^2} + C)\|\mathcal{A}_k(1;x) - 1\|_\infty \right. \\ &\quad + \frac{4Ch}{\delta^2}\|\mathcal{A}_k(t;x) - x\|_\infty \\ &\quad \left. + \frac{2C}{\delta^2}\|\mathcal{A}_k(t^2;x) - x^2\|_\infty \right) \\ &\leq H\left(\|\mathcal{A}_k(1;x) - 1\|_\infty + \|\mathcal{A}_k(t;x) - x\|_\infty + \|\mathcal{A}_k(t^2;x) - x^2\|_\infty\right), \end{aligned} \tag{2.23}$$

where $H = \max\{\epsilon + \frac{2Ch^2}{\delta^2} + C, \frac{4Ch}{\delta^2}, \frac{2C}{\delta^2}\}$.
Let take

$$\mathcal{L}_k(.;x) = \frac{1}{R_k}\sum_{v=0}^{k} p_v q_v \left\{ \frac{1}{(1+q)^v} \sum_{r=0}^{v} \binom{v}{r} q^{v-r} \mathcal{A}_r(.;x) \right\} \tag{2.24}$$

Now replacing $\mathcal{A}_k(.;x)$ by $\mathcal{L}_k(.;x)$ in both sides of (2.23), we get

$$\|\mathcal{L}_k(f;x) - f(x)\|_\infty < H(\|\mathcal{L}_k(1;x) - 1\|_\infty + \|\mathcal{L}_k(t;x) - x\|_\infty + \|\mathcal{L}_k(t^2;x) - x^2\|_\infty)$$

For $\epsilon' > 0$, let

$$E = \left\{ k \in \mathbb{N} : \|\mathcal{L}_k(1;x) - 1\|_\infty + \|\mathcal{L}_k(t;x) - x\|_\infty + \|\mathcal{L}_k(t^2;x) - x^2\|_\infty \geq \frac{\epsilon'}{H} \right\}$$

$$\begin{aligned} E_1 &= \left\{ k \in \mathbb{N} : \|\mathcal{L}_k(1;x) - 1\|_\infty \geq \frac{\epsilon'}{3H} \right\} \\ E_2 &= \left\{ k \in \mathbb{N} : \|\mathcal{L}_k(t;x) - x\|_\infty \geq \frac{\epsilon'}{3H} \right\} \\ E_3 &= \left\{ k \in \mathbb{N} : \|\mathcal{L}_k(t^2;x) - x^2\|_\infty \geq \frac{\epsilon'}{3H} \right\} \end{aligned}$$

Now

$$E \subseteq E_1 \cup E_2 \cup E_3 \tag{2.25}$$

Hence

$$\delta(E) \subseteq \delta(E_1) \cup \delta(E_2) \cup \delta(E_3) \tag{2.26}$$

Now using conditions (2.8)-(2.10), we get

$$(\overline{N}_{p_n q_n}.E^q)(st) - \lim_{k\to\infty} \|\mathcal{A}_k(f;x) - f(x)\|_\infty = 0. \tag{2.27}$$

This completes the proof of the theorem.

III. Order of statistical $(\overline{N}, p_n, q_n)(E,q)$-summability

In this section, we study the rate of statistical $(\overline{N}, p_n, q_n)(E,q)$ summability for a sequence of positive linear operators $(\mathcal{A}_k)$ defined on $\mathbb{C}[a,b]$. Here we begin by presenting the following definition.

Definition 1: Let (a_m) be a positive non-increaseing sequence. Then the sequence $u = (u_k)$ is said to be statistical $(\overline{N}, p_n, q_n)(E,q)$ summable to 'l' with the rate $o(a_m)$ if for every $\epsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{a_m}\left|\left\{ n \leq m : |T_n^{\overline{N}E} - l| \geq \epsilon \right\}\right| = 0 \tag{3.28}$$

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In this case, we write $u_k - l = (\overline{N}_{p_n q_n}.E^q)(st) - o(a_m)$.

By using the above definition we get the following lemma.

Lemma 1: Let (a_m) and (b_m) be two positive non-inceasing sequences. Let $u = (u_k)$ and $v = (v_k)$ be two sequences such that $u_k - l_1 = (\overline{N}_{p_n q_n}.E^q)(st) - o(a_m)$ and $v_k - l_2 = (\overline{N}_{p_n q_n}.E^q)(st) - o(b_m)$. Then,

(i)$\alpha(u_k - l_1) = (\overline{N}_{p_n q_n}.E^q)(st) - o(a_m)$, for any scalar α.

(ii)$(u_k - l_1) \pm (v_k - l_2) = (\overline{N}_{p_n q_n}.E^q)(st) - o(c_m)$, where $c_m = \max\{a_m, b_m\}$.

(iii) $(u_k - l_1).(v_k - l_2) = (\overline{N}_{p_n q_n}.E^q)(st) - o(a_m b_m)$.

Now recall the notions of modulus of continuity of f in $\mathbb{C}[a,b]$ is defined as

$$\omega(f,\delta) = \sup\{|f(t) - f(y)| : t, y \in [a,b], |t - y| < \delta\}$$

Hence, we get

$$\|f(t) - f(y)\| \leq \omega(f,\delta)\left\{\frac{|t-y|}{\delta} + 1\right\}$$

Next, we will prove the following theorem.

Theorem 3: Let $(\mathcal{A}_k)$ be a sequence of positive linear operators such that $\mathcal{A}_k : \mathbb{C}[a,b] \to \mathbb{C}[a,b]$ satisfies the following conditions

(i)$\|\mathcal{A}_k(1;x) - 1\|_\infty = (\overline{N}_{p_n q_n}.E^q)(st) - o(a_m)$

(ii)$\omega(f,\delta_k) = (\overline{N}_{p_n q_n}.E^q)(st) - o(b_m)$, where $\delta_k(x) = \sqrt{(\mathcal{A}_k(\varphi^2;x))}$ and $\varphi(x) = (t - x)$,

where (a_m) and (b_m) are positive non-increasing sequences. Then for all $f \in \mathbb{C}[a,b]$ and $c_m = \max\{a_m, b_m\}$, we have

$$\|\mathcal{A}_k(f;x) - f(x)\|_\infty = (\overline{N}_{p_n q_n}.E^q)(st) - o(c_m) \tag{3.29}$$

Proof: Let $f \in \mathbb{C}[a,b]$ for all $x \in [a,b]$. Then using modulus of continuity, the equation (2.18) can be reformed into

$$\begin{aligned}
|\mathcal{A}_k(f;x) - f(x)| &\leq \mathcal{A}_k\left(|f(t) - f(x)|, x\right) + |f(x)||\mathcal{A}_k(1;x) - 1| \\
&\leq \mathcal{A}_k\left(1 + \frac{|t-x|}{\delta}; x\right)\omega(f,\delta) + |f(x)||\mathcal{A}_k(1;x) - 1| \\
&\leq \mathcal{A}_k\left(1 + \frac{(t-x)^2}{\delta^2}; x\right)\omega(f,\delta) + |f(x)||\mathcal{A}_k(1;x) - 1| \\
&\leq \left(\mathcal{A}_k(1;x) + \frac{\mathcal{A}_k(\varphi^2;x)}{\delta^2}\right)\omega(f,\delta) + |f(x)||\mathcal{A}_k(1;x) - 1| \\
&\leq \omega(f,\delta)|\mathcal{A}_k(1;x) - 1| + |f(x)||\mathcal{A}_k(1;x) - 1| + \omega(f,\delta) \\
&\quad + \omega(f,\delta)\frac{1}{\delta^2}\mathcal{A}_k(\varphi^2;x)
\end{aligned} \tag{3.30}$$

If we choose $\delta = \delta_k = \sqrt{(\mathcal{A}_k(\varphi^2;x))}$, we get

$$\begin{aligned}
\|\mathcal{A}_k(f;x) - f(x)\|_\infty &\leq \|f\|_\infty\|\mathcal{A}_k(1;x) - 1\|_\infty + \omega(f,\delta_k)\|\mathcal{A}_k(1;x) - 1\|_\infty + 2\omega(f,\delta_k) \\
&\leq \lambda\left\{\|\mathcal{A}_k(1;x) - x\|_\infty + \omega(f,\delta_k)\|\mathcal{A}_k(1;x) - 1\|_\infty + \omega(f,\delta_k)\right\},
\end{aligned} \tag{3.31}$$

where $\lambda = \max\{\|f\|_\infty, 2\}$.

Now replacing $\mathcal{A}_k(.;x)$ by $\mathcal{L}_k(.;x)$ in (3.31), we get

$$\|\mathcal{L}_k(f;x) - f(x)\|_\infty \leq \lambda\{\|\mathcal{L}_k(1;x) - 1\|_\infty + \omega(f,\delta_k) + \omega(f,\delta_k)\|\mathcal{L}_k(1;x) - 1\|_\infty\} \tag{3.32}$$

Now, using Definition 1 and conditions (i) and (ii) of Lemma 1, we get

$$\|\mathcal{A}_k(f;x) - f(x)\|_\infty = (\overline{N}_{p_n q_n}.E^q)(st) - o(c_m), \tag{3.33}$$

which completes the proof of the theorem.

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