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# ADVANCED MATHEMATICS MATRICES REVIEWING RUTISHAUSER AND LANCZOS METHODS

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**ABSTRACT**

*In engineering, Eigen value problems are commonly encountered. Since there is heavy demand for solving complicated problems, research in area of Numerical linear algebra is very active which is associated with stability and perturbation analysis for practical application. In recent year, a number of method have been proposed for finding Eigen values such as Power, Lanczos, Jacobi, Rutishauser etc. The work of H. Rutishauser proved to be fundamental in many areas of numerical analysis, especially, algorithms for solving Eigen value problems. The method of Lanczos has become one of the most successful methods for approximating a few Eigen values of a real symmetric matrix. In this paper the existing method available has been explained, where the original matrix is reduced to a tridiagonal matrix whose Eigen values are same as those of original matrix. The intention of this paper is just to review the methods and give step by step procedure to compute Eigen values.*

***Keywords: Eigen values, Householder method, Lanczos method, Matrices, Rutishauser method.***

## I. INTRODUCTION

Many Eigen value problems are challenging in themselves and scientific computing depends critically in one way or another on numerical linear algebra algorithm. A single algorithm which always gives a very efficient solution of generalized Eigen value problem does not exist. Various methods are available for the solution of their problems, each are having its own particular advantages and disadvantages. This paper sketches advanced mathematics method currently in use to obtain an efficient solution of a specific Eigen value problem. An effort is made to highlight the two methods-1] Rutishauser Method and 2] Lanczos Method , which seemed most important for computing Eigenvalues. In Rutishauser method, QR algorithm is an Eigen value algorithm: that is, a procedure to calculate the Eigen values and Eigen vectors of a matrix. The basic idea is to perform QR decomposition, writing the matrix as a product of an orthogonal matrix and an upper triangular matrix, multiply the factors in reverse order and iterate. Lanczos method is used for obtaining good approximation to the smallest eigenvalue for generalized eigenvalue problem.

## II. RUTISHAUSER METHOD

1. A very precise method for real symmetric positive definite matrices is Rutishauser Method.

2. H. Rutishauser was a successful pioneer in numerical analysis at beginning of computer age.
3. Iterative methods for finding all Eigenvalues of non-symmetric matrices have appeared only recently.
4. Rutishauser uses QR method where Q is orthogonal and R is upper triangular.
5. In this method we first transform a given matrix A into tridiagonal matrix using Householder Method.
6. The Householder Method creates many zeros and thus reduces the further amount of work.
7. The problem reduces then to finding Eigen values of a tridiagonal form.
8. We compute $B_1$, $B_2$ stepwise according to following iteration method or Rotation.

**Step 1)** [1st Rotation]

Factor $B_K = Q_K R_K$

Where $Q_K \rightarrow$ Orthogonal

$R_K \rightarrow$ Upper triangular

Then compute $B_{K+1} = R_K Q_K$

**Step 2)** [2nd Rotation]

Factor $B_{K+1} = Q_{K+1} R_{K+1}$

Then compute $B_{K+2} = R_{K+1} Q_{K+1}$ and so on

## 2.1 Methodolgy

CONSIDER

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mj} & a_{mn} \end{bmatrix}$$

WHERE i=1, 2… m

j= 1, 2… n

### 2.1.1 HOUSEHOLDER'S METHOD

$$S = \sqrt{(a_{12})^2 + (a_{13})^2 + \cdots + (a_{1n})^2}$$

$$V_{K+1} = \sqrt{\frac{1}{2}\left[1 + \frac{a_{12}}{S}(SIGN\ OF\ a_{12})\right]}$$

$$V_{K+2} = \frac{a_{13}[SIGN\ OF\ a_{12}]}{2V_{K+1}\,S}$$

$$V_{K+3} = \frac{a_{14}[SIGN\ OF\ a_{12}]}{2V_{K+2}\,S}$$

$$V = [0\ V_{K+1}\ V_{K+2}\ V_{K+3}\ldots\ldots]$$

Where K=1, 2, 3, 4…

$$P = [I - 2\,V\,V^T]$$

$$P=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1-2(V_{K+1})^2 & -2V_{K+1}V_{K+2} & 0\\ 0 & -2V_{K+1}V_{K+2} & 1-2(V_{K+2})^2 & -2V_{K+2}V_{K+3}\\ 0 & 0 & -2V_{K+2}V_{K+3} & 1-2(V_{K+3})^2\end{bmatrix}$$

### 2.1.2 RUTISHAUSER METHOD

For simplifying we will consider $A_{3*3}$ Matrix

$$A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

$$P=\begin{bmatrix}1 & 0 & 0\\ 0 & 1-2(V_{K+1})^2 & -2V_{K+1}V_{K+2}\\ 0 & -2V_{K+1}V_{K+2} & 1-2(V_{K+2})^2\end{bmatrix}$$

$$B_K = P\ A\ P=\begin{bmatrix}b_{k11} & b_{k12} & b_{k13}\\ b_{k21} & b_{k22} & b_{k23}\\ b_{k31} & b_{k31} & b_{k33}\end{bmatrix}$$

$$C_K=\begin{bmatrix}\cos\theta_K & \sin\theta_K & 0\\ -\sin\theta_K & \cos\theta_K & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$C_{K+1}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_{K+1} & \sin\theta_{K+1}\\ 0 & -\sin\theta_{K+1} & \cos\theta_{K+1}\end{bmatrix}$$

**Equation [I]:**

$$[ROW\ 2\ OF\ C_K * COLUMN\ 1\ OF\ B_K]$$

$$-sin\theta_K * b_{k11} + cos\theta_K * b_{k21} = 0$$

$$\theta_K = tan^{-1}\frac{b_{k21}}{b_{k11}}$$

$$cos\theta_K = x_1$$

$$sin\theta_K = y_1$$

$$C_K=\begin{bmatrix}x_1 & y_1 & 0\\ -y_1 & x_1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$D=C_K B_K=\begin{bmatrix}x_1 & y_1 & 0\\ -y_1 & x_1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}b_{k11} & b_{k12} & b_{k13}\\ b_{k21} & b_{k22} & b_{k23}\\ b_{k31} & b_{k32} & b_{k33}\end{bmatrix}$$

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$$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

**Equation [II]:**

$$[ROW\ 3\ OF\ C_{K+1} * COLUMN\ 2\ OF\ D]$$

$$-sin\theta_{K+1} * d_{22} + cos\theta_{k+2} * d_{32} = 0$$

$$tan\theta_{K+1} = \frac{d_{32}}{d_{22}}$$

$$\theta_{K+1} = tan^{-1}\left[\frac{d_{32}}{d_{22}}\right]$$

$$cos\theta_{K+1} = x_2$$

$$sin\theta_{K+1} = y_2$$

$$C_{K+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x_2 & y_2 \\ 0 & -y_2 & x_2 \end{bmatrix}$$

$$R_K = [C_{K+1}][C_K][B_K]$$

$$R_K = \begin{bmatrix} r_{k11} & r_{k12} & r_{k13} \\ r_{k21} & r_{k22} & r_{k23} \\ r_{k31} & r_{k31} & r_{k33} \end{bmatrix}$$

$$Q_K = C_K^T C_{K+1}^T$$

$$= \begin{bmatrix} Q_{k11} & Q_{k12} & Q_{k13} \\ Q_{k21} & Q_{k22} & Q_{k23} \\ Q_{k31} & Q_{k32} & Q_{k33} \end{bmatrix}$$

$$B_{K+1} = R_K Q_K$$

$$B_{K+1} = \begin{bmatrix} b_{(k+1)^{11}} & b_{(k+1)^{12}} & b_{(k+1)^{13}} \\ b_{(k+1)^{21}} & b_{(k+1)^{22}} & b_{(k+1)^{23}} \\ b_{(k+1)^{31}} & b_{(k+1)^{32}} & b_{(k+1)^{33}} \end{bmatrix}$$

Which will be symmetric and tridiagonal.

The diagonal elements are eigenvalues and if off diagonal entries in $B_{K+1}$ are still larger in absolute values, go for the second iteration

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## III. LANCZOS METHOD

1. The Lanczos algorithm is a direct algorithm devised by Cornelius Lanczos that is an adaptation of power methods to find the most useful Eigen values and Eigen vectors of $m^{th}$ order linear system with a limited number of operations n.
2. Initially the method did not receive much attention because it was perceived as method of tridiagonalizing a matrix, a task that has better achieved by Givens and householders methods. To compete in accuracy, Lanczos method had to be supplemented with explicit orthogonalization of the computed vectors.
3. Many years later a renewed interest in the Lanczos method was caused by Paige's work, resulting in better understanding of method and widening its usability.
4. The Solution of the symmetric generalized Eigen value problem $A\phi = \lambda B\phi$ , Where A and B are real symmetric matrices is of significant practical important, especially in structural engineering as vibration problem and buckling problem.
5. In Lanczos Method, the matrix A is transformed into a tridiagonal matrix [T] in n steps in exact arithmetic, given a unit-normalized initial vector $X_k$.

### 3.1 Methodology

Let [A] [ϕ] = [λ̄ ] [B] [ϕ] be the generalized problem

$$\begin{bmatrix} \alpha_1 & \beta_2 & 0 & 0 & 0 & 0 \\ \beta_2 & \alpha_2 & \beta_3 & 0 & 0 & 0 \\ 0 & \beta_3 & \alpha_3 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \beta_{m-1} & 0 \\ 0 & 0 & 0 & \beta_{m-1} & \alpha_{m-1} & \beta_m \\ 0 & 0 & 0 & 0 & \beta_m & \alpha_m \end{bmatrix} [\phi] = \lambda \begin{bmatrix} b_{n1} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{n2} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{n3} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{n4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & b_n \end{bmatrix} [\phi]$$

Where A and B are real symmetric matrices

For simplification we will consider 5x5 matrix

$$\text{Where A} = \begin{bmatrix} \alpha_1 & \beta_2 & 0 & 0 & 0 \\ \beta_2 & \alpha_2 & \beta_3 & 0 & 0 \\ 0 & \beta_3 & \alpha_3 & \beta_4 & 0 \\ 0 & 0 & \beta_4 & \alpha_4 & \beta_5 \\ 0 & 0 & 0 & \beta_5 & \alpha_5 \end{bmatrix}$$

$$\text{B} = \begin{bmatrix} b_{n1} & 0 & 0 & 0 & 0 \\ 0 & b_{n2} & 0 & 0 & 0 \\ 0 & 0 & b_{n3} & 0 & 0 \\ 0 & 0 & 0 & b_{n4} & 0 \\ 0 & 0 & 0 & 0 & b_{n5} \end{bmatrix}$$

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$$X_K = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$ {Initial Vector Automatic Normalized}

WHERE K=1,2,3……

$$C = [\{X_K\}^T B\{X_K\}]^{\frac{1}{2}}$$

$$X_{K+1} = \frac{[X_K]^T}{C} = [x_{K+1}I \; x_{K+1}II \; x_{K+1}III \; x_{K+1}IV \; x_{K+1}V]$$

$$\bar{X}_{K+2} = \{A^{-1}\}\{B\}\{X_{K+1}\}$$

$$A^{-1} = \frac{adjA}{|A|}$$

$$\bar{X}_{K+2} = \begin{bmatrix} \bar{x}_{K+2}I \\ \bar{x}_{K+2}II \\ x_{K+2}III \\ \bar{x}_{K+2}IV \\ \bar{x}_{K+2}V \end{bmatrix}$$

$$\varepsilon_{K+1} = \left\{\bar{X}_{K+2}\right\}^T \{B\}\{X_{K+1}\}$$ Where $\varepsilon_{K+1}$ is scalar

$$\eta_{K+2} = \left[\left\{\bar{X}_{K+2}\right\}^T \{B\}\left\{\bar{X}_{K+2}\right\}\right]^{\frac{1}{2}}$$ Where $\eta_{K+2}$ is scalar

$$X_{K+2} = \frac{\bar{X}_{K+2}}{\eta_{K+2}} = \begin{bmatrix} x_{K+2}I \\ x_{K+2}II \\ x_{K+2}III \\ x_{K+2}IV \\ x_{K+2}V \end{bmatrix}$$

$$\bar{X}_{K+3} = \{A^{-1}\}\{B\}\{X_{K+2}\} = \begin{bmatrix} \bar{x}_{K+3}I \\ \bar{x}_{K+3}II \\ \bar{x}_{K+3}III \\ \bar{x}_{K+3}IV \\ \bar{x}_{K+3}V \end{bmatrix}$$

$$\varepsilon_{X+2} = \left\{\bar{X}_{K+3}\right\}^T \{B\}\{X_{K+2}\}$$

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$$T = \begin{bmatrix} \varepsilon_{X+1} & \eta_{X+2} \\ \eta_{X+2} & \varepsilon_{X+2} \end{bmatrix}$$

FOR EIGEN VALUES

$|T - \omega I| = 0$ Where $\omega = 1/\lambda$

$$\begin{vmatrix} \varepsilon_{x+1} - \omega & \eta_{x+2} \\ \eta_{x+2} & \varepsilon_{x+2} - \omega \end{vmatrix} = 0 \rightarrow [\text{MAT I}]$$

$$\omega_1 => \lambda_1 = 1/\omega_1$$

$$\omega_2 => \lambda_2 = 1/\omega_2$$

Putting Values of $\omega_1$ & $\omega_2$ respectively in MAT I to get Eigen vectors ɜ .

$$[T - \omega I]\phi = 0$$

$$\begin{bmatrix} \varepsilon_{x+1} - \omega_1 & \eta_{x+2} \\ \eta_{x+2} & \varepsilon_{x+2} - \omega_1 \end{bmatrix} \begin{bmatrix} ɜ_1 \\ ɜ_2 \end{bmatrix} = 0$$

Thus we obtain Eigenvalues ƛ 1, ƛ 2 and Eigen vectors ɜ .

## IV. APPLICATIONS

1. Construction of airplane
2. In control system and computer applications.
3. In structural engineering as vibration and buckling problems.
4. Spring-Mass system.

## V. CONCLUSION

Modern information processing deals with increasingly large matrices and efficient methods for computing eigenvalues and eigenvectors are extremely necessary. In most Eigenvalue problem, a single algorithm which always gives a very efficient solution does not exists. The paper describes solution methods which are currently in use and probably most efficient than others. If one of these methods is used appropriately, an efficient solution of a specific eigenvalue problem can be obtained. For large generalized eigenvalue problem arising from the two structural engineering applications, the vibration and buckling problem , the lanczos algorithm can be used. The paper has provided the Iterative methods in most simple and step by step form, which can form the basis of modern day eigenvalue computations.

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